Les équations différentielles (ch.1 à 3)

Exercice 1. (3+5=8 pts)

Série A

a) $\dfrac{dy}{dx} = \dfrac{x}{\cos(y)}$

$\Rightarrow \cos(y)\ dy = x\ dx$

$\Rightarrow \sin(y) = \dfrac{1}{2}\,x^2 + a\ ,a \in \mathbb{R} \Rightarrow$

$\Rightarrow \boxed{y(x) = \arcsin\left(\dfrac{1}{2}\,x^2 + a\right)\ ,a \in \mathbb{R}}$

b) $(E)\ y' - (2x+2)\cdot y = x+1\ ,\ y(0) = 3/2$

• $(E_h)\ y' - (2x+2)\cdot y = 0$

$\Rightarrow \dfrac{1}{y}\ dy = (2x+2)\ dx$

$\Rightarrow \ln|y| = x^2 + 2x + k\ ,k \in \mathbb{R}$

$\Rightarrow y_h(x) = a \cdot \mathrm{e}^{x^2+2x}\ ,a \in \mathbb{R}$

• méthode de la variation de la constante

$y_p(x) = c(x)\cdot \mathrm{e}^{x^2+2x}$

$\Rightarrow y_p'(x) = c'(x)\cdot \mathrm{e}^{x^2+2x} + c(x)\cdot(2x+2)\cdot \mathrm{e}^{x^2+2x}$

$\Rightarrow c'(x)\cdot \mathrm{e}^{x^2+2x} = x+1$

$\Rightarrow c'(x) = (x+1)\cdot \mathrm{e}^{-(x^2+2x)}$

$\Rightarrow c(x) = -\frac{1}{2}\cdot \mathrm{e}^{-(x^2+2x)} + a\ ,a \in \mathbb{R}$

$\Rightarrow y_p(x) = -\frac{1}{2} + a\cdot \mathrm{e}^{x^2+2x}\ ,a \in \mathbb{R}$

• $y(0) = \frac{3}{2} \Rightarrow a = 2 \Rightarrow$

$\Rightarrow \boxed{y(x) = -\dfrac{1}{2} + 2\cdot \mathrm{e}^{x^2+2x}}$

Série B

a) $\dfrac{dy}{dx} = -\dfrac{x}{\sin(y)}$

$\Rightarrow -\sin(y)\ dy = x\ dx$

$\Rightarrow \cos(y) = \dfrac{1}{2}\,x^2 + a\ ,a \in \mathbb{R} \Rightarrow$

$\Rightarrow \boxed{y(x) = \arccos\left(\dfrac{1}{2}\,x^2 + a\right)\ ,a \in \mathbb{R}}$

b) $(E)\ y' - (2x+4)\cdot y = x+2\ ,\ y(0) = 5/2$

• $(E_h)\ y' - (2x+4)\cdot y = 0$

$\Rightarrow \dfrac{1}{y}\ dy = (2x+4)\ dx$

$\Rightarrow \ln|y| = x^2 + 4x + k\ ,k \in \mathbb{R}$

$\Rightarrow y_h(x) = a \cdot \mathrm{e}^{x^2+4x}\ ,a \in \mathbb{R}$

• méthode de la variation de la constante

$y_p(x) = c(x)\cdot \mathrm{e}^{x^2+4x}$

$\Rightarrow y_p'(x) = c'(x)\cdot \mathrm{e}^{x^2+4x} + c(x)\cdot(2x+4)\cdot \mathrm{e}^{x^2+4x}$

$\Rightarrow c'(x)\cdot \mathrm{e}^{x^2+4x} = x+2$

$\Rightarrow c'(x) = (x+2)\cdot \mathrm{e}^{-(x^2+4x)}$

$\Rightarrow c(x) = -\frac{1}{2}\cdot \mathrm{e}^{-(x^2+4x)} + a\ ,a \in \mathbb{R}$

$\Rightarrow y_p(x) = -\frac{1}{2} + a\cdot \mathrm{e}^{x^2+4x}\ ,a \in \mathbb{R}$

• $y(0) = \frac{5}{2} \Rightarrow a = 3 \Rightarrow$

$\Rightarrow \boxed{y(x) = -\dfrac{1}{2} + 3\cdot \mathrm{e}^{x^2+4x}}$

Exercice 2. (5+2=7 pts)

a) • $T' = k \cdot (T-25)$; $T(0)=75$, $T(8)=40$

$\Rightarrow \dfrac{dT}{dt} = k \cdot (T-25)$, $k \in \mathbb{R}$

$\Rightarrow \dfrac{dT}{T-25} = k \cdot dt$

$\Rightarrow \ln|T-25| = kt + c$, $c \in \mathbb{R}$

$\Rightarrow |T-25| = e^{kt} \cdot e^{c}$, $c \in \mathbb{R}$

$\Rightarrow T-25 = a \cdot e^{kt}$, $a \in \mathbb{R}$

$\Rightarrow T(t) = a \cdot e^{kt} + 25$, $a \in \mathbb{R}$

ou méthode de la variation de la constante

ou méthode des coefficients indéterminés

• $T(0) = a + 25 = 75 \Rightarrow a = 50$

• $T(8) = 50 \cdot e^{8k} + 25 = 40 \Rightarrow k = \dfrac{\ln(3/10)}{8}$

$$\Rightarrow \boxed{T(t) = 50 \cdot \left(\frac{3}{10}\right)^{\frac{t}{8}} + 25}$$

b) $50 \cdot \left(\dfrac{3}{10}\right)^{\frac{t}{8}} + 25 = 30 \Rightarrow$

$\Rightarrow \left(\dfrac{3}{10}\right)^{\frac{t}{8}} = \dfrac{1}{10} \Rightarrow$

$$\Rightarrow \boxed{t = \frac{8 \cdot \ln(1/10)}{\ln(3/10)}} \ (\cong 15.3 \text{ minutes})$$

• $T' = k \cdot (T-25)$; $T(0)=70$, $T(7)=40$

$\Rightarrow \dfrac{dT}{dt} = k \cdot (T-25)$, $k \in \mathbb{R}$

$\Rightarrow \dfrac{dT}{T-25} = k \cdot dt$

$\Rightarrow \ln|T-25| = kt + c$, $c \in \mathbb{R}$

$\Rightarrow |T-25| = e^{kt} \cdot e^{c}$, $c \in \mathbb{R}$

$\Rightarrow T-25 = a \cdot e^{kt}$, $a \in \mathbb{R}$

$\Rightarrow T(t) = a \cdot e^{kt} + 25$, $a \in \mathbb{R}$

ou méthode de la variation de la constante

ou méthode des coefficients indéterminés

• $T(0) = a + 25 = 70 \Rightarrow a = 45$

• $T(7) = 45 \cdot e^{7k} + 25 = 40 \Rightarrow k = \dfrac{\ln(1/3)}{7}$

$$\Rightarrow \boxed{T(t) = 45 \cdot \left(\frac{1}{3}\right)^{\frac{t}{7}} + 25}$$

$45 \cdot \left(\dfrac{1}{3}\right)^{\frac{t}{7}} + 25 = 30 \Rightarrow$

$\Rightarrow \left(\dfrac{1}{3}\right)^{\frac{t}{7}} = \dfrac{1}{9} \Rightarrow$

$$\Rightarrow \boxed{t = \frac{7 \cdot \ln(1/9)}{\ln(1/3)}} \ (= 14 \text{ minutes})$$

Exercice 3. (5 pts)

$(E) \quad m \cdot v' + k \cdot v = m \cdot g \quad , \quad v(o) = v_0$

$\Rightarrow v' + \dfrac{k}{m} \cdot v = g \ , k, m, g \in \mathbb{R}$

• $(E_h) \ v' + \dfrac{k}{m} \cdot v = 0$

$\Rightarrow \dfrac{1}{v} \, dv = -\dfrac{k}{m} \, dt$

$\Rightarrow \ln |v| = -\dfrac{k}{m} \cdot \, t + c \, , c \in \mathbb{R}$

$\Rightarrow v_h(t) = a \cdot \mathrm{e}^{-\frac{k}{m} \cdot t} \, , a \in \mathbb{R}$

• méthode de la variation de la constante
(ou méthode des coefficients indéterminés avec $v_p(t) = \alpha$)

$v_p(t) = c(t) \cdot \mathrm{e}^{-\frac{k}{m} \cdot t}$

$\Rightarrow v_p'(t) = c'(t) \cdot \mathrm{e}^{-\frac{k}{m} \cdot t} + c(t) \cdot \left(-\dfrac{k}{m}\right) \cdot \mathrm{e}^{-\frac{k}{m} \cdot t}$

$\Rightarrow c'(t) \cdot \mathrm{e}^{-\frac{k}{m} \cdot t} = g$

$\Rightarrow c'(t) = g \cdot \mathrm{e}^{\frac{k}{m} \cdot t}$

$\Rightarrow c(t) = \dfrac{mg}{k} \cdot \mathrm{e}^{\frac{k}{m} \cdot t} + b \ , b \in \mathbb{R}$

$\Rightarrow v_p(t) = b \cdot e^{-\frac{k}{m} \cdot t} + \dfrac{mg}{k} \ , b \in \mathbb{R}$

• $v(0) = v_0 \Rightarrow b = v_0 - \dfrac{mg}{k}$

$\Rightarrow \boxed{v(t) = \left(v_0 - \dfrac{mg}{k}\right) \cdot e^{-\frac{k}{m} \cdot t} + \dfrac{mg}{k}}$